# Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

**August 2017: Problem 2 Solution**

**Exercise.** Let $X$ be a set, and $\mu^*$ be an outer measure on $X$ such that $\mu^*(X) < \infty$. Define $\nu^*$ by letting

$$\nu^*(E) = \sqrt{\mu^*(E)} \qquad \text{for every } E \subseteq X$$

Prove that

**(a)** $\nu^*$ is an outer measure

Solution.

$\nu^*$ is an **outer measure** if $\nu^* : \mathcal{P}(X) \to [0, \infty]$

i) $\nu^*(\emptyset) = 0$

ii) $\nu^*(A) \leq \nu^*(B)$ for $A \subseteq B$

iii) $\nu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \nu^*(A_j)$

Since $\mu^*$ is an outer measure $\mu^* \geq 0 \implies \nu^*(E) = \sqrt{\mu^*(E)} \geq 0$ for all $E \subseteq X$

i) Also, $\mu^*(\emptyset) = 0 \implies \nu^*(\emptyset) = \sqrt{\mu^*(\emptyset)} = \sqrt{0} = 0$

ii) For $A \subseteq B$,

$$\begin{aligned} \nu * (A) &= \sqrt{\mu^*(A)} \\ &\leq \sqrt{\mu^*(B)} && \text{since } 0 \leq \mu^*(A) \leq \mu^*(B) \\ &= \nu^*(B) \end{aligned}$$

iii)

$$\begin{aligned} \nu^*\left(\bigcup_1^\infty A_j\right) &= \sqrt{\mu^*\left(\bigcup_1^\infty A_j\right)} \\ &\leq \sqrt{\sum_1^\infty \mu^*(A_j)} \\ &\leq \sum_1^\infty \sqrt{\mu^*(A_j)} \\ &= \sum_1^\infty \nu^*(A_j) \end{aligned}$$

Thus, $\nu^*$ is an outer measure.

**(b)** A subset $A \subseteq X$ belongs in the Caratheodory $\sigma-$algebra of $\nu^*$ (the $\sigma-$algebra of $\nu^*-$measurable sets) if and only if $\nu^*(A) = 0$ or $\nu^*(A^C) = 0$.

Solution.

$A \subseteq X$ is **$\nu^*$ measurable** if $\nu^*(E) = \nu^*(E \cap A) + \nu^*(E \cap A^C)$ for all $E \subset X$.
A $\sigma-$algebra is closed under countable unions and complements.
($\Longleftarrow$) Since $\nu^*$ an outer measure,

$$\nu^*(E) = nu^*((E \cap A) \cup (E \cap A^C)) \leq \nu^*(E \cap A) + \nu^*(E \cap A^C)$$

And $(E \cap A) \subseteq E$ and $(E \cap A^C) \subseteq E$, so
If $\nu^*(A) = 0$

$$\nu^*(E) \leq \nu^*(E \cap A) + \nu^*(E \cap A^C) = \nu^*(E \cap A^C) \leq \nu^*(E)$$

If $\nu^*(A^C) = 0$

$$\nu^*(E) \leq \nu^*(E \cap A) + \nu^*(E \cap A^C) = \nu^*(E \cap A) \leq \nu^*(E)$$

Thus, $\nu^*(E) = \nu^*(E \cap A) + \nu^*(E \cap A^C)$ and $A$ is $\nu^*$ measurable and belongs to a Caratheodory $\sigma-$algebra of $\nu^*$
($\Longrightarrow$) If $A$ is in the $\sigma-$algebra of $\nu^*$ measurable sets then

$$\begin{aligned}
& \nu^*(E) - \nu^*(E \cap A) + \nu^*(E \cap A^C), \qquad \forall E \subseteq X \\
\Longrightarrow \quad & \sqrt{\mu^*(E)} = \sqrt{\mu^*(E \cap A)} + \sqrt{^*(E \cap A^C)} \\
\Longrightarrow \quad & \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) + \sqrt{\mu^*(E \cap A)\mu^*(E \cap A^C)} \\
& \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)
\end{aligned}$$

But $\mu^*$ is an outer measure so,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

Thus,

$$\mu^*(E \cap A)\mu^*(E \cap A^C) = 0$$

Setting $E = X$, this means $\mu^*(A) = 0$ or $\mu^*(A^C) = 0$